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ON ONE-SIDED POLYNOMIAL L^1 APPROXIMATION

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ABSTRACT. In this paper we derive one-sided L^1 approximation by polynomials of a certain function defined on interval $[-1,1]$.

1. INTRODUCTION

For a function f defined on segment $[a,b]$ and satisfying certain conditions a problem of one-sided L^1 approximation (see, e.g. [1], [2], [3], [4]) consists in finding polynomials P_n and p_n defined on $[a,b]$ of degree n such that $p_n(x) \leq f(x) \leq P_n(x)$ on $[a,b]$ and such that the integral $\int_a^b \frac{P_n(x)-p_n(x)}{\sqrt{(x-a)(b-x)}}dx$ is bounded by a certain function $g(n)$ which decays to zero, as $n \to +\infty$.

In applications of one-sided L^1 approximation to deriving theorems of Tauberian type for Laplace transform with optimal remainder term it is also important to control the growth of coefficients of polynomials P_n and p_n as well as the difference $P_n(x) - p_n(x)$, uniformly in $x \in [a,b]$.

In this paper we prove a theorem on one-sided L^1 approximation of function

$$F_{m,\beta}(x) := \begin{cases} \left[\ln\left(1+\frac{1-x}{1+x}\right)\right]^{\beta}\left[\ln\left(\frac{1+x}{1+b}\right)\right]^{m}, & \text{for } b \leq x \leq 1 \\ 0, & \text{for } -1 \leq x < b \end{cases} \tag{1.1}$$

by polynomials P_n and p_n of degree less than or equal to n such that the difference $P_n(x) - p_n(x)$ is bounded by $\frac{1}{n^m}$. Here m is non-negative integer, β is a real number which satisfies certain conditions and $b \in (-1,1)$ is fixed number.

One-sided L^1 approximation of function $F_{m,\beta}$ is important for applications to Tauberian type theorems for Laplace-Stieltjes transform (see, e.g. [6] where this function is used).

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2. Preliminaries

The techniques developed by Freud in [1], [2], [3], [4] and used in the proof of Tauberian type theorems for Laplace transforms are based on one-sided L^1 approximation by polynomials. Here we recall Freud's one-sided polynomial L^1 approximation theorem of m times differentiable function, whose mth derivative is of bounded variation.

Theorem 2.1. [5, Chapter VII, Theorem 7.1] [4, Theorem 1] *Let f on $[-1,1]$ be real and an indefinite integral of order $m \geq 1$ of a function of bounded variation. Then there are constants A_1, A_2 and A_3 such that for every $n \in \mathbf{N}$ there are polynomials p_n and P_n of degree $\leq n$ with the following properties:*

$$p_n(x) \leq f(x) \leq P_n(x), \quad -1 \leq x \leq 1; \tag{2.1}$$

$$\int_{-1}^{1} \frac{P_n(x) - p_n(x)}{\sqrt{1-x^2}} dx \leq \frac{A_1}{n^{m+1}}; \tag{2.2}$$

$$v(p_n), v(P_n) \leq A_2^n, \tag{2.3}$$

where $v(p_n)$ and $v(P_n)$ denote sums of absolute values of coefficients of polynomials p_n and P_n respectively.

J. Korevaar [5] applied Freud's approximation Theorem 2.1 to the function

$$F_m(x) := \begin{cases} \left[\ln\left(\frac{1+x}{1+b}\right)\right]^m, & \text{za } b \leq x \leq 1; \\ 0, & \text{za } -1 \leq x < b \end{cases} \tag{2.4}$$

where $b \in (-1,1)$ is fixed number, to prove the theorem:

Theorem 2.2. [5, Chapter VII, Theorem 8.1] *For an integer $m \geq 0$ and fixed $b \in (-1,1)$, let $F_{m,\beta}(x)$ be the function given by* (2.4)*. Then there are constants D_1, D_2 and D_3 such that for every integer $k \geq 2\mu + 1$ there are polynomials q_k and Q_k of degree $\leq k$ with the following properties:*

$$q_k(x) \leq F_m(x) \leq Q_k(x) \text{ on } (-1,1);$$

$$Q_k(-1) = q_k(-1) = 0,\ Q_k(1) = q_k(1) = 1;$$

$$\int_{-1}^{1} [Q_k(x) - q_k(x)] \frac{dx}{(1-x^2)^{\mu+1/2}} \leq \frac{D_1}{k^{m+1}};$$

$$v(q_k), v(Q_k) \leq D_2 D_3^k.$$

Theorem 2.2. is important for derivation of the Tauberian type theorem for the Laplace transform, see [5, Chapter VII, Theorem 3.1 and Theorem 3.2]. In certain applications, specially in number theory (see [6]) it is of interest to derive Tauberian type theorems for the Laplace-Stieltjes transform, in which case a polynomial L^1-approximation of the function (1.1) which is generalisation of (2.4) is needed.

3. Main result

In the proof of our main result we need the following lemma.

Lemma 3.1. *Let the real function f, defined on $[-1,1]$ be indefinite integral of order $m \geq 1$ of some function $\overline{f_m}(x)$ of bounded variation. Then there exist constants A_1, A_2 and A_3 such that for $n \in \mathbf{N}$ large enough there are polynomials p_n and P_n of degree less than or equal to n with the following properties:*

$$p_n(x) \leq f(x) \leq P_n(x), \quad -1 \leq x \leq 1; \tag{3.1}$$

$$\int_{-1}^{1} \frac{P_n(x) - p_n(x)}{\sqrt{1-x^2}} dx \leq \frac{A_1}{n^{m+1}}; \tag{3.2}$$

$$v(p_n), v(P_n) \leq A_2^n; \tag{3.3}$$

$$P_n(x) - p_n(x) \leq \frac{A_3}{n^m}, \quad -1 < x < 1, \tag{3.4}$$

where $v(p_n)$ and $v(P_n)$ denote sums of absolute values of coefficients of polynomials p_n and P_n respectively.

Proof. First three properties follow from Theorem 2.1. Assume that the property (3.4) is not true. Then, for every positive number A inequality

$$P_n(x) - p_n(x) > \frac{A}{n^m}, \quad x \in (-1,1), \text{ holds for } n \in \mathbf{N} \text{ large enough.}$$

It implies that

$$\int_{-1}^{1} \frac{P_n(x) - p_n(x)}{\sqrt{1-x^2}} dx > \frac{A}{n^m} \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{A\pi}{n^m} = \frac{An\pi}{n^{m+1}},$$
$$\forall A > 0, \qquad \text{for } n \in \mathbf{N} \text{ large enough.}$$

This contradicts the property (3.2). Therefore, we may conclude that the property (3.4) hold. □

We are interested in approximation of function $F_{m,\beta}(x)$ given by (1.1) where m is non-negative integer, β is real number which satisfies certain conditions and $b \in (-1,1)$ is fixed number. The following Theorem generalizes the result of Theorem 2.2.

Theorem 3.2. *For non-negative integers m and μ, real number $\beta \geq m + \mu$ and a fixed $b \in (-1,1)$, let $F_{m,\beta}(x)$ be the function given by* (1.1)*. Then, there are constants D_1, D_2, D_3 and D_4 such that for integers $n \geq 2\mu + 1$ large enough there are polynomials p_n and P_n of degree less than or equal to n with the following properties:*

$$p_n(x) \leq F_{m,\beta}(x) \leq P_n(x), \textit{ for } x \in (-1,1);$$

$$\int_{-1}^{1} [P_n(x) - p_n(x)] \frac{dx}{(1-x^2)^{\mu+1/2}} \leq \frac{D_1}{n^{m+1}};$$

$$\nu\left(p_n\right), \nu\left(P_n\right) \le D_2 D_3^n;$$
$$P_n(x) - p_n(x) \le D_4/n^m, \ \textit{for} \ \ x \in (-1,1).$$

Proof. For given integer $\mu \ge 1$ we will consider the function

$$f(x) = \frac{F_{m,\beta}(x)}{(1-x^2)^{\mu}}. \tag{3.5}$$

Around $x = 1$, write

$$F_{m,\beta}(x) = \left(\frac{1-x}{1+x}\right)^{\beta} G_{m,\beta}(x),$$

where $G_{m,\beta}$ has Taylor series expansion of every order around $x = 1$. Near the point $x = 1$ we have

$$f(x) = \frac{(1-x)^{\beta-\mu}}{(1+x)^{\beta+\mu}} G_{m,\beta}(x).$$

The function $f(x)$ is an indefinite integral of order m of a function of bounded variation on $(-1,1)$, since $\beta > m+\mu$.

Applying Lemma 3.1 to the function $f(x)$ defined by (3.5) we deduce that there exist constants A_1, A_2 and A_3 such that for every integer n greater or equal to $2\mu+2$ there are polynomials $p^*_{n-2\mu}$ and $P^*_{n-2\mu}$ of degree less than or equal to $n-2\mu-1$ such that the following four properties are satisfied:

$$p^*_{n-2\mu}(x) \le f(x) \le P^*_{n-2\mu}(x), \ \ \text{for} \ \ -1 \le x \le 1;$$
$$\int_{-1}^{1} \left[P^*_{n-2\mu}(x) - p^*_{n-2\mu}(x)\right] \frac{dx}{\sqrt{1-x^2}} \le \frac{A_1}{(n-2\mu-1)^{m+1}};$$
$$\nu(p^*_{n-2\mu}), \nu(P^*_{n-2\mu}) \le A_2^{n-2\mu-1};$$
$$P^*_{n-2\mu}(x) - p^*_{n-2\mu}(x) \le \frac{A_3}{(n-2\mu-1)^m}, \ \text{for} \ \ -1 < x < 1.$$

Now, we can define polynomials p_n and P_n by:

$$p_n(x) = \left(1-x^2\right)^{\mu} p^*_{n-2\mu}(x), \quad P_n(x) = \left(1-x^2\right)^{\mu} P^*_{n-2\mu}(x).$$

Since

$$\nu\left((1-x^2)^{\mu}\right) = 2^{\mu},$$

using above properties of a functions $p^*_{n-2\mu}$ and $P^*_{n-2\mu}$ we have

$$\frac{p_n(x)}{(1-x^2)^{\mu}} \le f(x) \le \frac{P_n(x)}{(1-x^2)^{\mu}} \ \ \text{for} \ \ -1 \le x \le 1;$$
$$\int_{-1}^{1} \frac{P_n(x) - p_n(x)}{(1-x^2)^{\mu}} \frac{dx}{\sqrt{1-x^2}} \le \frac{A_1}{(n-2\mu-1)^{m+1}};$$
$$\nu(p_n), \nu(P_n) \le 2^{\mu} A_2^{n-2\mu-1};$$
$$\frac{P_n(x) - p_n(x)}{(1-x^2)^{\mu}} \le \frac{A_3}{(n-2\mu-1)^m}, \ \ \text{for} \ \ -1 < x < 1.$$

By definiton of a function f we have

$$p_n(x) \le F_{m,\beta}(x) \le P_n(x) \ \text{ for } \ -1 \le x \le 1;$$
$$\int_{-1}^{1} [P_n(x) - p_n(x)] \frac{dx}{(1-x^2)^{\mu+1/2}} \le \frac{A_1}{(n-2\mu-1)^{m+1}};$$
$$\nu(p_n), \nu(P_n) \le 2^{\mu} A_2^{n-2\mu-1};$$
$$P_n(x) - p_n(x) \le \frac{A_2}{(n-2\mu-1)^m} \ \text{ for } \ -1 < x < 1.$$

Then, there are constants D_1, D_2, D_3 and D_4 such that for integers $n \ge 2\mu+1$ large enough the following four properties are satisfied:

$$p_n(x) \le F_{m,\beta}(x) \le P_n(x); \ \text{for} \ -1 \le x \le 1;$$
$$\int_{-1}^{1} [P_n(x) - p_n(x)] \frac{dx}{(1-x^2)^{\mu+1/2}} \le \frac{D_1}{n^{m+1}};$$
$$\nu(p_n), \nu(P_n) \le D_2 D_3^n;$$

$$P_n(x) - p_n(x) \le D_4/n^m; \ \text{ for } \ -1 < x < 1.$$

This completes the proof. □

Remark 3.3. Theorem 3.2 is an improvement of Theorem 2.2 in the sense that it treats more general function $F_{m,\beta}$ defined by (1.1). Theorem 3.2 also gives control of the growth of the difference $P_n(x) - p_n(x)$ as follows $P_n(x) - p_n(x) \le D_4/n^m$, for $x \in (-1,1)$.

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